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The Global Optimization of Variational Problems with Discontinuous Solutions

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Abstract. We consider variational problems in which the slope of the admissible curves is not necessarily bounded, so that they admit discontinuous solutions. A problem is first reformulated as one consisting of the minimization of an integral in a space of functions satisfying a set of integral equalities; this is then transfered to a nonstandard framework, in which Loeb measures take the place of the functions and a near-minimizer can always be found. This is mapped back to the standard world by means of the standard part map; its image is a minimizer, so that the optimization is *global*. The minimizer is shown to be the solution of an infinite dimensional linear program and by well-proven approximation procedures a finite dimensional linear program is found by means of which nearly-optimal curves can be constructed for the original problem. A numerical example is given.

Key words: Global optimization, Calculus of variations, Discontinuous solutions, Nonstandard analysis, Loeb measures, Standard part map.

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1. Introduction

We have developed in many publications (see Rubio, 1986, 1994 and the references there) an approach to the study to the global optimization of nonlinear optimal control problems based on the consideration of measure spaces and related mathematical structures; this approach was suggested by the work of Young (1969) on the calculus of variations. In most of our previous work, all underlying sets — control sets being a special case — were taken to be compact.

In this paper we consider a simple variational problem whose control set — the set in which the slopes of the admissible curves take values — is noncompact, unbounded; we have already extended our results to such a case — in a different manner — some years ago (Rubio, 1976). These problems have interest because they include as possible minimizing curves those exhibiting discontinuities; perhaps the first thorough semiclassical treatment of such problems was by Lawden (1959); one could also mention Krotov (1961). Nowadays optimal control problems exhibiting 'impulses' in their control law have been much studied, see Vinter and Pereira (1988), Bressan and Rampazzo (1993) as well as their references; see also our own treatment in Rubio (1994, Chapter 6).

Our general philosophy is somewhat different from the one prevailing in most of those references, including our own work (Rubio, 1994). It is always necessary in circumstances such as those found in this paper to introduce idealized elements, to complete — in a very general sense — the natural spaces arising in connection with the optimization problems. So far, it has appeared natural to enlarge the spaces to include 'delta functions', impulses; there are many ways of doing this, such as embedding the spaces into spaces of distributions, using nonstandard versions of this same construction, and so on. Alas, it is very difficult to work with impulses; in particular, it is very hard to define functions of impulses; see Rubio (1994, Chapter 6), for a discussion of this point. In this paper, our idealized elements lack the familiarity of impulses and such; they are *nonstandard elements*, not easily visualized but easily handled mathematically.

Thus, our path is as follows. The variational problem will be written in a manner involving the solution of a set of integral equalities; these are mapped then into a nonstandard framework, in which the use of Loeb measures gives rise to an important result, that a near-minimizer for the standard optimization problem always exists. The standard part map provided us with a *global minimizer* for the original problem, as well as with a measure-theoretical framework in the standard world in which a linear program is obtained with the minimizer as a solution.

Approximation tools developed in our previous work (Rubio, 1986) are then used to develop a finite dimensional approximation of the linear program, and construct nearly-optimal solutions of the variational problem. A numerical example is given.

2. The problem

Let x, z be vectors in Euclidean n-space $\mathbb{R}^n$, t a real variable, $J := [t_a, t_b]$ with $t_a < t_b$, A a compact subset of $\mathbb{R}^n$, x_a, x_b points in A. Consider the class $\mathcal{F}$ of infinitely differentiable functions $t \to x(t), t \in J$ such that

$$x(t) \in A, t \in J, \quad x(t_a) = x_a, x(t_b) = x_b.$$

Note that there are no constraints on the slope $\dot{x}$ of these admissible curves. We assume that this class $\mathcal{F}$ is nonempty, and seek to minimize the functional $I : \mathcal{F} \to \mathbb{R}$

$$I(x(\cdot)) = \int_{t_a}^{t_b} f_0(t, x(t), \dot{x}(t))\, dt, \tag{1}$$

for $x(\cdot) \in \mathcal{F}$. Here $(t, x, z) \to f_0(t, x, z)$ is a continuous function defined on

$$\Omega := J \times A \times \mathbb{R}^n$$

Of course, in general there will not be a minimizer for the functional I in $\mathcal{F}$— this class, of infinitely differentiable functions, is not large enough. The question that arises now is, what can we enlarge this class to, while keeping the functional

I well defined in the new class? The problem seems to be that there are cases in which the infimum occurs at a function which has discontinuities, see Lawden (1959) and Rubio (1976); what can then be the meaning of the derivative $\dot{x}$, and of the integral in the definition (1) of I? These are questions that will be answered by the constructions to be introduced in this paper; for the moment, we take this definition of the class $\mathcal{F}$ as a starting point, a temporary device.

We develop nows some equalities that are satisfied by the admissible curves, and which will serve as the key tool for our nonstandard treatment. Let B be an open ball in $\mathbb{R}^{n+1}$ containing $J \times A$; we denote by $C'(B)$ the space of all real-valued functions on B that are uniformly continuous on B together with their first derivatives. Let $\phi \in C'(B)$; define

$$\hat{\phi}(t,x,z) := \phi_x(t,x)z + \phi_t(t,x) \tag{2}$$

for all $(t,x,z) \in \Omega$. Of course, $\hat{\phi} \in C(\Omega)$. If $x(\cdot)$ is an admissible curve,

$$\begin{aligned} \int_J \hat{\phi}(t,x(t),\dot{x}(t))\,dt \;&=\; \int_J [\phi_x(t,x(t))\dot{x}(t) + \phi_t(t,x(t))]\,dt \\ &= \int_J \dot{\phi}(t,x(t))\,dt = \phi(t_b,x_b) - \phi(t_a,x_a) := \Delta\phi, \end{aligned} \tag{3}$$

for all $\phi \in C'(B)$. There are two special cases which are of interest; in the first we put $\psi(t,x) := x_j\psi(t)$, with $1 \le j \le n$, and ψ a test function on the interior J° of the interval J; that is, $\psi \in \mathcal{D}(J^\circ)$; see Rubio (1986). Then, putting

$$\psi_j(t,x,z) := x_j\psi'(t) + z_j\psi(t), \tag{4}$$

for $1 \le j \le n$ and $\psi \in \mathcal{D}(J^\circ)$, the equality (3) becomes

$$\int_J \psi_j(t,x(t),\dot{x}(t))\,dt = 0, \tag{5}$$

since the test functions in $\mathcal{D}(J^\circ)$ are zero at the boundary of J.

The second case of interest happens when the function ϕ is chosen as a differentiable function of the time t only,

$$\psi(t,x,z) := \theta(t), (t,x,z) \in \Omega, \tag{6}$$

then $\hat{\psi}(t,x,u) = \dot{\theta}(t)$, $(t,x,z) \in \Omega$. We introduce a subspace of $C(\Omega)$, to be denoted by $C_1(\Omega)$, consisting of those functions which depend only on the first variable t; then the equalities (3) become:

$$\int_J h(t,x(t),\dot{x}(t))\,dt = a_h, \quad h \in C_1(\Omega), \tag{7}$$

with a_h the Lebesgue integral of $h(\cdot,x,z)$ over J, independent of x and z.

As explained in detail in some of our previous publications (Rubio, 1976, 1986), we will choose for each of these spaces countable sets of functions whose linear combinations are dense in the corresponding spaces in appropriate topologies.

For the space $C'(\Omega)$ we shall choose $\{\phi_i\}$; a set of polynomials in $(t, x_1, \ldots, x_n)$; for $\mathcal{D}(J^\circ), \{\chi_j\}$, the sequence of functions of the type (4) when the functions ψ are the sine and cosine functions

$$\sin(2\pi r(t - t_a)/\Delta t)), \quad 1 - \cos(2\pi r(t - t_a)/\Delta t)), r = 1, 2, \ldots$$

and $j = 1, \ldots, n$, and for $C_1(\Omega)$ the sequence $\{h_k\}$, a set of monomials in t (but watch how we use in Section 5 pulse-like, lower semicontinuous functions instead). Further, we shall consider a finite number of the resulting infinite number of equalities, as in

$$\begin{aligned} \int_J \hat{\phi}_i(t, x(t), \dot{x}(t))\, dt &= \Delta\phi_i, i = 1, \ldots, M_1, \\ \int_J \chi_j(t, x(t), \dot{x}(t))\, dt &= 0, j = 1, \ldots, M_2, \\ \int_J h_k(t, x(t), \dot{x}(t))\, dt &= a_{h_k}, k = 1, \ldots, M_3, \end{aligned} \tag{8}$$

and study the properties of the curves satisfying them. Then we shall take limits, as $M_1, M_2, M_3 \to \infty$.

3. The nonstandard way

We shall change our framework here — in a manner that appears minor. Let $\overline{\mathbb{R}}$ be the extended real line. Instead of assuming that the slopes of the curves in $\mathcal{F}$ take values in $\mathbb{R}^n$, we shall take $\overline{\mathbb{R}}^n$ as a place of abode for these values. There will be no apparent change — the curves do take values in $\mathbb{R}^n$ and $\mathbb{R}^n \subset \overline{\mathbb{R}}^n$. But, as we shall see below, the introduction of $\overline{\mathbb{R}}$ is fundamental to out development. We consider therefore the problem of minimizing the functional

$$I(x(\cdot)) := \int_J f_0(t, x(t), \dot{x}(t))\, dt \tag{9}$$

of the class $\mathcal{F}_M$ of C^∞ functions valued in $\mathbb{R}^n$ satisfying

$$\int_J f_i(t, x(t), \dot{x}(t))\, dt = b_i, i = 1, \ldots, M, \tag{10}$$

and taking values in $A \subset \mathbb{R}^n$. Here $f_0, f_i, i = 1, \ldots, M$, are in $C(\Omega')$, with

$$\Omega' := J \times A \times \overline{\mathbb{R}}^n.$$

We shall write sometimes $w := (t, x, z) \in \Omega'$. The integer $M \geq 1$ is fixed, and so are the constant $b_i, i = 1, \ldots, M$. We assume that the class $\mathcal{F}_M$ is nonempty. We shall develop in this section a procedure to enlarge the set $\mathcal{F}_M$, while at the same time extending the functional (9) to the whole of the new, larger set of admissible elements. This procedure will be based on nonstandard techniques.

The reason why this is so involves the fact that the slope of the curves in $\mathcal{F}_M$ is not necessarily bounded, and it may happen, for some integrands, that a minimizer for the functional I exhibits jump-type discontinuities (Lawden, 1959), at which points the slope $\dot{x}$ could be said to be infinite. Thus, we have to be able to handle infinity — or, actually, infinities, a task for which nonstandard analysis is well suited.

In our quest for infinities, we shall start with the extended real line $\overline{\mathbb{R}}$. This will be part of our starting nonstandard construction, while also playing a major role when we return to the standard world. We will review briefly some of its properties; see Berge (1963), Monroe (1953) and Choquet (1969).

- The extended real line $\overline{\mathbb{R}}$ is obtained by adding to the real line $\mathbb{R}$ two elements, ∞ and $-\infty$, so that $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty, -\infty\}$. These two elements satisfy a number of well-known postulates, such as
  - For every $x \in \mathbb{R}$, $-\infty < x < \infty$. This makes the extended real line into an ordered set.
  - The extended system will not be a field, but we can connect the new elements with the field operations by postulating that for every real number x we have:
  $$x/\pm\infty = 0;\ (\pm\infty)(\pm\infty) = \infty;\ \infty + \infty + x = \infty,$$
  etc.
- It is possible to put a topology on $\overline{\mathbb{R}}$ so that it is a *compact space*. Such a topology is generated by the following sets:
  - The open sets in $\mathbb{R}$.
  - The union of $\{\infty\}$ with an open set of $\mathbb{R}$ containing an interval of the form (λ, ∞).
  - The union of $\{-\infty\}$ with an open set of $\mathbb{R}$ containing an interval of the form $(-\infty, \lambda)$.

We proceed now with our nonstandard construction. For general treatments of this topic, see Cutland (1988) and Rubio (1994). We will work in a nonstandard framework given by a superstructure $V(W)$, $\overline{\mathbb{R}} \subset W$. The superstructure $V({}^*V)$ is also an enlargement, and $\aleph_1$-saturated. We study integrals of the form (9,10), that is,

$$\int_J f(t, x(t), \dot{x}(t))\, dt, \tag{11}$$

with $x(\cdot) \in \mathcal{F}_M$ and $f \in C(\Omega')$. Then,

$$(\forall x(\cdot) \in \mathcal{F}_M) \left(\int_J f(t, x(t), \dot{x}(t))\, dt \in \overline{\mathbb{R}} \right); \tag{12}$$

by transfer,

$$(\forall x(\cdot) \in^* \mathcal{F}_M) \left({}^*\int_{{}^*J} {}^*f(t, x(t), \dot{x}(t))\, dt \in {}^*\overline{\mathbb{R}} \right), \tag{13}$$

where here and below we write $\dot{x}(\cdot)$ for $({}^*\frac{d}{dt}x(\cdot))$. Thus, the nonstandard version of the optimization problem (9,10) consists in minimizing

$$ {}^*I(x(\cdot)) :={}^* \int_{{}^*J} {}^*f_0(t, x(t), \dot{x}(t))\, dt \tag{14} $$

on the class ${}^*\mathcal{F}_M$ of functions $x(\cdot) \in {}^*C^{\infty n}(\Omega')$, $x(t) \in {}^*A, t \in {}^*J$, satisfying

$$ {}^* \int_{{}^*J} {}^*f_i(t, x(t), \dot{x}(t))\, dt = b_i, i = 1, \ldots, M. \tag{15} $$

Consider now the map suggested by (12). If $x(\cdot) \in \mathcal{F}_M$ is fixed, the map

$$ \nu_{x(\cdot)} : F \to \int_J F(t, x(t), \dot{x}(t))\, dt \in \overline{\mathbb{R}}, F \in C(\Omega') \tag{16} $$

is linear and positive. By Riesz' Theorem, there is a measure, to be denoted also by $\nu_{x(\cdot)}$, on the Borel sets $\mathcal{B}$ of Ω', that represents this map; remember that Ω' is compact. Then $({}^*\Omega', {}^*\mathcal{B}, {}^*\nu_{x(\cdot)})$ is a nonstandard measure space and then (Render, 1993),

LEMMA 1. *There is a measure space* $({}^*\Omega', \mathcal{A}, \mu_L^{x(\cdot)})$ *so that* $\mu_L^{x(\cdot)}$ *is the Loeb measure associated with* $\nu_{x(\cdot)}$*; then,*

$$ {}^* \int_{{}^*J} F(t, x(t), \dot{x}(t))\, dt = \mu_L^{x(\cdot)}(F) := \int_{{}^*\Omega'} F d\, \mu_L^{x(\cdot)}, F \in C({}^*\Omega'). \tag{17} $$

The algebra $\mathcal{A}$ *is an extension of the algebra* ${}^*\mathcal{B}$.

Proof. Follows directly from the reference given above. □

Thus, one can write the optimization problem (14, 15) as the problem of minimizing

$$ J(\mu_L^{x(\cdot)}) := \mu_L^{x(\cdot)}(f_0), \tag{18} $$

over the set $\mathcal{M}_M^L$ of measures of the form $\mu_L^{x(\cdot)}$ defined by

$$ \mu_L^{x(\cdot)}(f_i) = b_i, i = 1, \ldots, M. \tag{19} $$

The following two propositions show that the solution of our problem is a *global optimizer*.

PROPOSITION 1. *(i) The infima associated with the problems* $(9-10)$, $(14-15)$ *and* $(18-19)$ *are equal.*

(ii) For any positive infinitesimal $s \in {}^*\overline{\mathbb{R}}$, *we can find a* near-minimizer $\mu_s \in \mathcal{M}_M^L$ *for the functional* J *in (18) in the set* $\mathcal{M}_M^L$, *so that*

$$ J(\mu_s) = \inf_{\mathcal{M}_M^L} J + s. \tag{20} $$

Proof. It follows from Theorem 3.8 in Rubio (1994). □

Let, then, s be a fixed positive infinitesimal in $^*\overline{\mathbb{R}}$, and μ_s the corresponding near-minimzer for J on $\mathcal{M}_M^L$. We can proceed to map back this measure to the standard world, by means of the *standard part map*, see Henson (1979), Aldaz (1992), Render (1993) and Landers and Rogge (1987).

PROPOSITION 2. *There is a Baire measure μ_{opt} on Ω' so that:*

(i) If S is a Baire set in Ω',

$$\mu_{opt}(S) = {}^\circ\mu_s(\mathrm{st}_{\Omega'}^{-1}(S)),$$

where $\mathrm{st}_{\Omega'}^{-1}(S)$ is the union of the monads of the elements of S.

(ii)

$$\mu_{opt}(f_0) := \int_{\Omega'} f_0 d\,\mu_{opt} = \inf_{\mathcal{F}_M} \int_J f_0(t, x(t), \dot{x}(t))\,dt.$$

(iii) The measure μ_{opt} is a solution of the following optimization problem. Minimize

$$\mu(f_0) \tag{21}$$

over the set $\mathcal{M}_M^+(\Omega')$ of positive Baire measures on Ω' satisfying

$$\mu(f_i) = b_i, i = 1, \ldots, M. \tag{22}$$

(iv) The support of μ_{opt} contains subsets of Ω' in which at least one component of the variable $z \in \overline{\mathbb{R}}^n$ is either $-\infty$ or ∞. The measure μ_{opt} is defined by a Baire measure on $J \times A \times \mathbb{R}^n$ plus atomic measures on those subsets.

Proof. (i) See Henson (1979). (ii), (iii). These statements follow from Proposition 1 and the fact that for all $f \in C(\Omega')$

$$\int_{\Omega'} f\,d\,\mu_{opt} = \int_{^*\Omega'} {}^\circ(^*f)\,d\,\mu_s = {}^\circ\int_{^*\Omega'} {}^*f\,d\,\mu_s;$$

note that by continuity

$${}^\circ(^*f(w)) = f(\mathrm{st}_{\Omega'}(w)) = f(y_w), w \in \Omega,'$$

where y_w is the (unique) element of Ω' so that w is in the monad of y.

(iv) We consider now the support of μ_{opt}. For simplicity in the notation, assume $n = 1$, and consider a point $(t, x, \infty) \in S$, S being a Baire set in Ω'. Then

$$\mathrm{st}_{\Omega'}^{-1}(t, x, \infty) = M \times \bigcap_{\lambda} {}^*(\lambda, \infty] = M \times {}^*\{\infty\},$$

with M the monad of (t, x). Then, for $f \in C(\Omega')$, for some hyperreal α, there will be a contribution to the integral

$$\int_{^*\Omega'} {}^\circ(^*f)\,d\,\mu_s$$

of

$$^{\circ}[< \alpha_i f(t,x,\infty) >] = (^{\circ}\alpha) f(t,x,\infty),$$

which proves our contention; the cases including the element $-\infty$ and multidimensional vectors can be treated similarly. □

In problems of interest, in which the function f_0 tends to infinity at infinity, and in which the infimum is finite, elements $(t,x,z) \in \Omega'$ with one or more components of value ∞ or $-\infty$ do not really occur in the support of μ_{opt}; note that expressions such as $\infty - \infty$ are not defined for the extended real line. Thus,

PROPOSITION 3. *Suppose that*

$$|f_0(t,x,z)| = \infty$$

whenever a component of z is either ∞ or $-\infty$, and that the minimum associated with the linear program (21)–(22) is finite. Then such elements are not present in the support of μ_{opt}.

We are now in a strong position to solve our original problem — the optimization problem (9) and (10) in the standard world. Note that we have been able to construct an extension of the original space $\mathcal{F}_M$, made up of elements which are not curves; however, the action of μ_{opt} — a global optimizer — can be approximated by members of $\mathcal{F}_M$.

4. Approximation

We consider the optimization problem (21)–(22). By means of a result of Rosenbloom (1952), and since Ω' is compact, we can state that the minimizer μ_{opt} for this problem has the form

$$\mu_{opt} = \sum_{\ell=1}^{M} \alpha_\ell \delta(w_\ell), \alpha_\ell \geq 0, w_\ell \in \Omega', \ell = 1, \ldots, M, \tag{23}$$

where $\delta(w)$ is the atomic measure with support $\{w\} \in \Omega'$. It is appropriate now to write the system (21)–(22) in full, as in (8). We wish to minimize

$$\sum_{\ell=1}^{M} \alpha_\ell f_0(w_\ell), \tag{24}$$

on the set defined by the elements

$$\alpha_\ell \geq 0, w_\ell \in \Omega', \ell = 1, \ldots, M,$$

which satisfy, further,

$$\begin{aligned}\sum_{\ell=1}^{M}\alpha_\ell\hat{\phi}_i(t_\ell,x_\ell,z_\ell) &= \Delta\phi_i, i=1,\ldots,M_1,\\ \sum_{\ell=1}^{M}\alpha_\ell\chi_j(t_\ell,x_\ell,z_\ell) &= 0, j=1,\ldots,M_2,\\ \sum_{\ell=1}^{M}\alpha_\ell h_k(t_\ell,x_\ell,z_\ell) &= a_{h_k}, k=1,\ldots,M_3,\end{aligned} \tag{25}$$

with $M := M_1 + M_2 + M_3$.

A further concept must be introduced now; see Rubio (1986). Note that we have in (24)–(25) a *nonlinear* optimization problem, in which the unknowns are the coefficients α_ℓ and supports $w_\ell, \ell = 1,\ldots,M$. In order to find a linear approximation to this problem, we consider ω, a countable dense subset of Ω'. Taking $N \gg M$ elements from ω, including all elements of the form introduced in (iv) above in which at least one component of $z \in \overline{\mathbb{R}}^n$ is either $-\infty$ or ∞, we can write (24)–(25) as follows. We wish to minimize

$$\sum_{\ell=1}^{N}\alpha_\ell f_0(w_\ell), \tag{26}$$

on the set defined by the elements $\alpha_\ell \geq 0, \ell = 1,\ldots,M$, which satisfy, further,

$$\begin{aligned}\sum_{\ell=1}^{N}\alpha_\ell\hat{\phi}_i(t_\ell,x_\ell,z_\ell) &= \Delta\phi_i, i=1,\ldots,M_1,\\ \sum_{\ell=1}^{N}\alpha_\ell\chi_j(t_\ell,x_\ell,z_\ell) &= 0, j=1,\ldots,M_2,\\ \sum_{\ell=1}^{N}\alpha_\ell h_k(t_\ell,x_\ell,z_\ell) &= a_{h_k}, k=1,\ldots,M_3,\end{aligned} \tag{27}$$

with $M := M_1 + M_2 + M_3$. Here, then, the supports w_ℓ are fixed, in ω; the coefficients $\alpha_\ell, \ell = 1,\ldots,M$, are the only unknowns; this is an $M \times N$ (finite dimensional) linear program. Of course as $N \to \infty$ the support of the optimal measure μ_{opt} in (24)–(25) can be approximated closer and closer by that of μ_{opt}^N, the solution of (26)–(27). Note, further, that at most M of the unknown α's are nonzero; we shall assume that the problem has essential regularity, and that exactly M of these α's are nonzero; see Rubio (1986, Chapters 3 and 4), for a discussion of this point.

Since no element z_ℓ in the support of μ_{opt}^N has components equal to either $-\infty$ or ∞, the approximation process has been studied in detail in Rubio (1986), and a curve in $\mathcal{F}_M$ can be constructed approximating the action of μ_{opt} on f_0. $-\infty$ or ∞. It is necessary to modify the set ω into a set

$$\omega_Q := \omega \cap [-Q, Q],$$

in which the coordinates of the z direction takes values in a portion of the dense set ω defined by a number Q; if Q is large enough, all the elements in the support of μ_{opt} will be approximated adequately. Then,

PROPOSITION 4. *Suppose that the minimum in the linear program (26)–(27) is finite, that the conditions of Proposition 3 are satisfied, and that the function f_0 is Lipschitz, that is, that there is a constant k so that*

$$|f_0(t', x', z') - f_0(t, x, z)| \leq k(|t' - t| + \|x' - x\| + \|z' - z\|)$$

for all $(t', x', z'), (t, z, x)$ in Ω. Then it is possible to construct a curve $x(\cdot)^N_Q$ in $\mathcal{F}_M$ so that:

(i) As $Q \to \infty$,

$$\int_J f_0(t, x^N_Q(t), \dot{x}^N_Q(t))\, dt \to \mu^N_{opt}(f_0).$$

(ii) As $N \to \infty$,

$$\mu^N_{opt}(f_0) \to \mu_{opt}(f_0).$$

Proof. (i)Let μ^N_Q be the solution of the linear programming problem (26)–(27) with ω modified into ω_Q as explained above. Then,

1. The time set $J = [t_a, t_b]$ has been divided into M_3 equal subdivisions $J_k, k = 1, \ldots, M_3$, each of measure $\Delta t/M_3$, with $\Delta t := t_b - t_a$.
2. There is a total of M indices ℓ associated with those values α_ℓ that are nonzero. To each of the subdivision J_k of J defined above are associated a number of these indices; if only one is so associated, then the value of α_ℓ equals $\Delta t/M_3$; if more than one are associated, then the sum of the corresponding α_ℓ's adds up to $\Delta t/M_3$.
3. Without loss of generality, let J_k, for $1 \leq k \leq M_3$ be associated with two α_ℓ's, α_{ℓ_1} and α_{ℓ_2}, as explained above, a typical situation; then we build the curve $x(\cdot)^N_Q$ on J_k by, first, building the derivative curve $\dot{x}(\cdot)^N_Q$ by making $\dot{x}(t)^Q_N, t \in J_k$, equal to z_{ℓ_1} or z_{ℓ_2} in each of the two partitions of J_k with lengths α_{ℓ_1} and α_{ℓ_2} respectively. After building the whole of the curve $\dot{x}(\cdot)^N_Q$ on J, this curve can be integrated using the initial conditions and thus obtain $x(\cdot)^N_Q$, which satisfies exactly the final condition, as explained above.
4. We shall also build a further curve, $\breve{x}(\cdot)^N_Q$, constructed from the results of the linear program so that $\breve{x}^N_Q(t) = x_\ell$ on the corresponding subdivision of J_k.

5. Assume that α_q is associated with the subdivision J_p and, for simplicity and without loss of generality, that q is the only index associated with this subdivision, so that $\alpha_q = \Delta t / M_3$. Then the construction and approximation procedures indicated above takes place with no difficulty. Here we are comparing

$$\alpha_q f_0(t_q, x_q, z_q)$$

with

$$\int_{J_p} f_0(t, x_Q^N(t), \dot{x}_Q^N(t))\, dt;$$

the mth component of z_q is ∞, while the mth component of $\dot{x}_Q^N(t)$ equals Q on J_p. Thus:

$$\left| \alpha_q f_0(t_q, x_q, z_q) - \int_{J_p} f_0(t, x_Q^N(t), \dot{x}_Q^N(t))\, dt \right| \tag{28}$$

$$\leq \left| \alpha_q f_0(t_q, x_q, z_q) - \int_{J_p} f_0(t, \breve{x}_Q^N(t), \dot{x}_Q^N(t))\, dt \right| \tag{29}$$

$$+ \left| \int_{J_p} f_0(t, \breve{x}_Q^N(t), \dot{x}_Q^N(t))\, dt - \int_{J_p} f_0(t, x_Q^N(t), \dot{x}_Q^N(t))\, dt. \right| \tag{30}$$

By the mean value theorem

$$\int_{J_p} f_0(t, \breve{x}_Q^N(t), \dot{x}_Q^N(t))\, dt = \alpha_q f_0(\hat{t}, x_q, \hat{z}_q),$$

where $\hat{t} \in J_p$ and $\hat{z}_q$ equals z_q if Q is sufficiently large. Thus, the quantity in (29) is not higher than

$$\alpha_q (k(\hat{t} - t_q) + R_Q) \leq \alpha_q (\alpha_q k + R_Q),$$

where R_Q is zero if Q is sufficiently large. Thus the quantity associated with (29) tends to zero as M_3 tends to infinity, provided Q is large enough. The quantity associated with (30) has been treated in Rubio (1986, Chapter 4).

(ii) As in Rubio (1986, Chapters 3 and 4). □

It is interesting to realize that in some problems such as the one to be treated numerically below, the support of μ_{opt} gets wider as $M \to \infty$, so that larger and larger values of Q are necessary; in this way, solution curves with steep portions are obtained.

5. An example and discussion

We have solved a simple problem, taken from Lawden (1959), with $n = 1$ and

$$f_0(t, x, z) := (z^3 - 1)^{1/3};$$

the other parameters are $t_a = 0, t_b = 1$, $x_a = 0, x_b = 2$. The numerical approximation was performed with the following parameters:

$$Q = 24, M_1 = 2, M_2 = 8, M_3 = 30, M = 40, N = 27000.$$

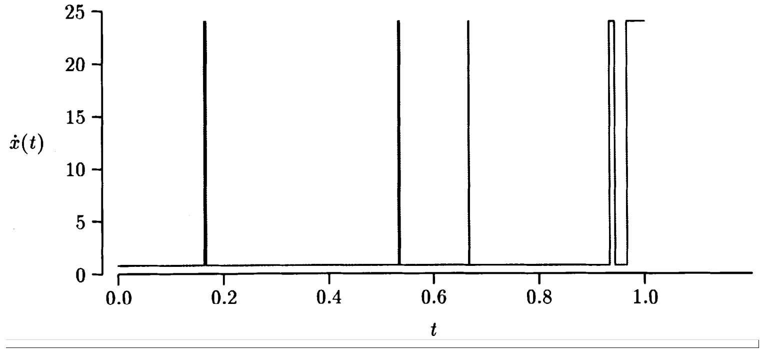

Figure 1. Graph of the slope $\dot{x}(\cdot)$ of the nearly-optimal curve $x(\cdot)$.

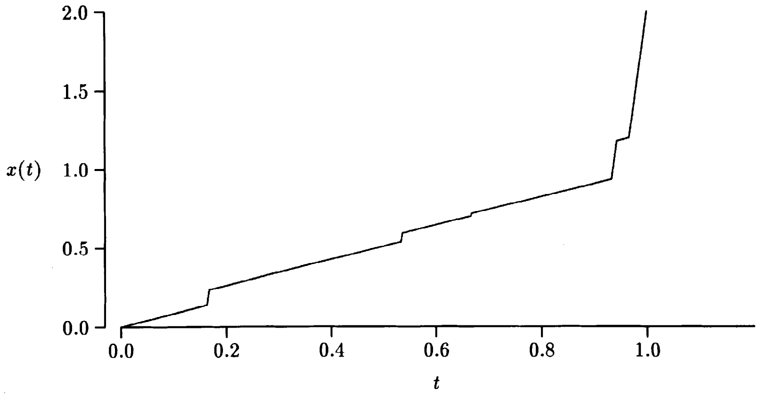

Figure 2. Graph of the nearly-optimal curve $x(\cdot)$.

The functions h_k has been chosen as pulse-like functions in t defined as follows. The t-axis is divided into M_3 subintervals, and the function h_k equals 1 in the kth subdivision, 0 elsewhere, the values given at the boundaries so that these functions are lower semicontinuous. The theory can be adapted very well to such choice, see Rubio (1986, Chapter 5).

Each of the axis associated with the variables (t, x, z) was divided into 30 parts; the minimum obtained was 0.49587, which should be compared with the minimum for the problem obtained by semiclassical means in Lawden (1959) of 0.413. The approximation problems for this kind of optimization problems are fierce; it is necessary to have a large value of Q as well as very fine mesh, thus very large linear programs. The graphs of the derivative $\dot{x}(\cdot)$ and the curve $x(\cdot)$ can be seen above.

We should note that — even in this simply problem — we have achieved something not easily accomplished by the more traditional methods, which would have found it extremely hard to deal with the cube of a ‘delta function’; they mostly deal in problems in which the slope variable — our z — or the control variable appear *linearly*.

The method employed here should generalize without much difficulty to more general optimal control problems with unbounded control sets. Also, it appears promising to deal with partial differential equations with solutions exhibiting shocks, such as those studied in Oberguggenberger (1992).

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